

PLASTICITY, COULOMB FRICTION AND SLIDING IN THE LIMIT ANALYSIS OF MASONRY ARCHES

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Abstract. *In the framework of plasticity theory, limit analysis in the presence of large value of friction is a consolidated approach to analyse the collapse of the arch and identify the corresponding collapse mechanism. Inclusion of Coulomb friction and sliding in the limit analysis, however, points out an extremely delicate question, since in accordance with the plastic theory, as first outlined by Drucker, for non-standard materials Coulomb friction results in a non-associated flow rule which invalidates the general bounding theorems. Thus, the limit load can be evaluated by means of more restrictive theorems. By adopting, for example, a normality rule, the correct limit load can be reached, even if it may result in incorrect failure modes since the normality rule is not obeyed by mechanisms involving Coulomb friction.*

Purpose of this paper is that of revisiting the bounding theorems of plasticity for masonry arches in the presence of finite Coulomb friction and sliding by applying and extending Drucker's and Radenkovic's theorems. It will be demonstrated that for a masonry arch subject to its own weight, although finite friction makes it locally possible activation of sliding characterized by a non-associated flow rule, corresponding mechanisms for the whole arch are kinematically not-admissible with the exception of the “mechanism for the arch of minimum-thickness with finite friction” in the range $0.395 \geq \mathbf{m} \geq 0.310$. This result gives a greater consistency to the statement that the collapse of the arch due to sliding in the presence of typical values of the friction coefficient is unlikely.

1 INTRODUCTION

A consolidated approach to investigate the behaviour of a masonry arch is that of modelling it as a system made of a no-tension rigid-plastic material characterized by infinite strength to compression. This model within the framework of the plastic theory was first proposed by Heyman^{1, 2}, who also assumed large values of the friction coefficient in order to prevent sliding.

Accordingly, the safety of a masonry arch - treated as a rigid-plastic curvilinear beam - depends on the existence of any thrust line within the thickness of the arch; therefore, the collapse condition corresponds to the formation of a pattern of “hinges” at the intrados and extrados, capable of transforming the statically indeterminate structure into a mechanism.

The key idea of Heyman’s^{1, 2} approach thus consists in an appropriate transfer of the basic methods of the plastic theory from the “steel” to the “stone skeleton”, since for standard materials, any limit analysis admits the same limit state for both rigid-plastic and elastic-plastic constitutive relationships.

Heyman’s^{1, 2} approach allowed also the reintroduction of the historical pre-elastic tradition started by La Hire³ in 1712 - with its “wedge theory” - and continued during eighteenth and beginning of nineteenth-century by Coulomb⁴, Mascheroni⁵ and Persy⁶; in accordance, the masonry arch can be modelled as a system of rigid voussoirs subject, at the contact joints, to extended unilateral constraints and Coulomb friction.

After Heyman’s work, significant advances in limit analysis of masonry arches were made taking into account the actual strength of the material (Livesley⁷, Delbecq⁸) and the possibility of sliding mechanisms (Livesley⁷, Gilbert⁹).

The inclusion of sliding in the presence of Coulomb friction, as first observed by Drucker¹⁰, invalidates the general bounding theorems of plasticity since for non-standard materials Coulomb friction results in a non-associated flow rule; thus, more restrictive criteria need to be applied. Drucker’s criteria, as will be better discussed in the following, state that the limit load of a system with frictional interfaces is bounded from below by the limit load of the same system with zero friction, and from above by the limit load for no relative sliding and also by the limit load for the same system cemented by a cohesion-less soil with dilatancy angle $\gamma \leq \arctan m$ where m is the Coulomb friction coefficient.

This last criterion was adopted by Gilbert⁹ to evaluate the limit load by means of the kinematic theorem and by adopting a normality rule ($\gamma = \arctan m$). However, though such an approach can give the correct value of the limit load, it results in an incorrect failure mode since normality is not obeyed by mechanisms involving Coulomb friction. For non-standard material, Livesley⁷ proposed, on the other hand, the use of the lower theorem and an evolutive method to determine both the limit state and corresponding collapse mode: in his approach, first, a normality rule is adopted to reach the sliding yield surface, then, the Coulomb dissipation rule is imposed by adding self-equilibrating solutions to identify new possible limit states. Among the identified mechanisms, the collapse one is that which does not require any dilatancy.

With respect to the same problem, the fact that the masonry arch can be modelled as a system made of rigid voussoirs - subject to Coulomb friction and to extended unilateral constraints at the contact joints - has suggested (Sinopoli¹¹) in the line traced by Heyman the revisitation of historical theories elaborated on the subject during eighteenth and beginning of nineteenth-century. Thus, a modern formulation in terms of virtual work theorem was

proposed to revisit and, eventually, to confirm the theory proposed by Mascheroni⁵, and the “maximis and minimis” method proposed by Coulomb⁴ and improved by Persy⁶. The revisitation of Mascheroni’s kinematic approach allowed to identify, for the masonry arch made of standard material, both the well-known five-hinges rotational collapse mode and the La Hire’s collapse mode with sliding joints at the haunches for zero friction (Sinopoli¹²); La Hire’s collapse mode corresponds to the Drucker’s lower bound for non-standard material. Moreover, the revisitation of Coulomb and Persy’s static approach allowed to identify the rotational collapse mode for standard material and a mixed collapse mode for non-standard material, which we will rename here *collapse mode of the arch of minimum thickness for finite friction*.

In this paper, the methods adopted in (Sinopoli¹¹) will be discussed in the framework of the plasticity theory for standard and non-standard materials; then, the results obtained will be verified. In Sec. 2, a brief recall of the bounding theorems will be made. In Sec. 3, the mechanical model of the arch and its constitutive relationship will be introduced. In Sec. 4 local and global yield surfaces of the arch made of standard material will be identified for values of Coulomb friction large enough to prevent sliding. Finite friction will be the subject of Sec. 5; Drucker and Radenkovic’s¹³ domains will be discussed and a modified Radenkovic’s domain will be proposed to identify safe regions of stability for the arch. The formulation in terms of virtual work for the kinematic theorem will enable us to identify, for the non-standard Drucker’s boundary of the arch, all the possible sliding collapse modes and the corresponding dissipated energy.

2 LOWER AND UPPER BOUND THEOREMS OF PLASTICITY

Consider an elastic perfectly-plastic material with yield condition: $F(\mathbf{s}_{ij}) = 0$. Elastic admissible states correspond to: $F(\mathbf{s}_{ij}) < 0$, while states for which $F(\mathbf{s}_{ij}) > 0$ are impossible. Let $G(\mathbf{s}_{ij})=0$ be the plastic potential function of the considered material. If $F \circ G$, then the material is called standard and a normality rule characterizes the flow rule and the associated limit state (Sacchi¹⁴). In this case, the power of the dissipation function depends only on the direction of the displacement field; it is always zero. Thus, bounding theorems hold which give the same limit state. They can be formulated as follows:

UPPER BOUND: If in a given situation, by means of any method, an unstable collapse mechanism can be found, then the elastic-plastic model of the structure is unstable.

LOWER BOUND: If in a given situation, by means of any method, a potentially stable stress state can be found, then the elastic-plastic model of the structure is stable.

If the limit state is evaluated as the critical value of a load parameter \mathbf{I} , thus, the critical load given by both static and kinematic theorems is the same.

On the contrary, if $F \neq G$ the material has a non-standard behaviour¹⁵; the power of the dissipation function – generally different from zero - depends not only on the direction of the displacement field, but also on the value of the limit stress state. In this case, the bounding theorems can be formulated only for corresponding standard materials. They become:

UPPER BOUND: Drucker’s Theorem for corresponding standard material. If in a given situation, by means of any method, an unstable collapse mechanism can be found for an identical structure made of standard material, then the non-standard elastic-plastic model of

the structure is unstable.

Drucker's domain for the corresponding standard material can be determined either statically from outside by identifying the stress states which are unstable for the whole structure or for a part of it, or kinematically from outside by identifying unstable mechanisms. In this last case, the instability condition for standard material corresponds to: $L^{(e)} \geq L^{(i)}$, where $L^{(e)}$ is the work performed by the external forces and $L^{(i)}$ is the internal work. Note that the critical load for the corresponding Drucker's standard material $I^{S,D}$ is either larger than or equal to the critical load $I^{NS,D}$ of the considered non-standard material.

LOWER BOUND: Radenkovic's Theorem. If, in a given situation, by means of any method, a potentially stable stress state can be found for an identical structure made of standard material to which the same dissipation power corresponds, then the non-standard elastic-plastic model of the structure is stable.

The boundary of the Radenkovic's domain is the envelope of planes perpendicular to the displacement field of the Drucker's boundary for the considered non-standard material. It is by this way that the dissipation power of both limit states - Radenkovic's and non-standard Drucker's yielding - is the same. It is obvious that the critical load for the Radenkovic's standard material $I^{S,R}$ is either lower than or equal to the corresponding critical load $I^{NS,R}$ for the considered non-standard material (Smars¹⁶).

Therefore, a gap exists between the non-standard Drucker's boundary and both the standard Drucker and Radenkovic's boundaries; inside the gap regions, potentially stable states exist, but their stability is not sure. Same considerations need to be made for non-standard rigid-plastic materials and, therefore, for the limit analysis of masonry arches.

3 THE ARCH AS A RIGID-PLASTIC SYSTEM

Consider a two-dimensional symmetric arch made of n rigid voussoirs and subject to its own weight, and examine the problem of its equilibrium.

Number the voussoirs as j , ranging from 1 to n , and the joints connecting two adjacent voussoirs as k , ranging from 1 to $n+1$. In addition, assume a local reference system for the j -th voussoir (Fig. 1), with origin at any point of the joint of the adjacent $j+1$ -th voussoir, with which a unit vector system (T, N) is associated; N is oriented outward.

Adopt a rigid-plastic constitutive relationship for no-tension material with infinite strength to compression and subject to Coulomb's friction; further, assume that the masonry arch can be completely represented by the family of its joints, identified by the corresponding values of angle α (Fig.1).

At any joint, the admissible stress states are those corresponding to the stress-resultant R applied at any point of the joint with extreme position either at the intrados or at the extrados. The normal and tangential component R^N and R^T of the resultant R are subject to the constraints:

$$R^N = R \cdot N \leq 0 \quad (1a)$$

$$|R^T| \leq m |R^N| \quad (1b)$$

As outlined by Delbecq⁸, the local stability analysis which can be consequently performed,

gives a stability condition weaker than that obtainable by considering the arch as a continuum and by identifying the admissible stress state at any point of the structure.

4 STABILITY FOR STANDARD BEHAVIOUR

If the friction coefficient is large enough to prevent sliding, a normality rule characterizes the yielding surface at any joint. However, the arch is a statically undetermined structure, so that the value of R^N is unknown. Nevertheless, the symmetry of both geometry and dead loads makes the resultant at the crown horizontal, that is, at the crown: $|R^N|=H$. Thus, the equilibrium at joint \mathbf{a} can be investigated as the corresponding equilibrium of the voussoir extending from the crown joint to angle \mathbf{a} (Fig. 1), by assuming the unknown thrust H - applied at any point of the crown joint - as a function of the parameter $K=R/r$, where R is the extrados radius and r the intrados radius of the arch.

The normality rule enables the structure to generate any value of H below the yielding boundary. Therefore, determine the Drucker's domain at joint \mathbf{a} from above, by considering extreme application points for H at the crown (namely, at the extrados and at the intrados) and extreme application points of \mathbf{R} at joint \mathbf{a} (again, at the extrados and at the intrados). Inequality (1a) gives the Drucker's domain at joint \mathbf{a} as shown in Fig. 2; it is the segment AB of negative slope \mathbf{a} completely internal to the Coulomb's domain. Points A and B define the yielding surface and the extreme values for R^N ; they correspond to the lowest and largest values of H , respectively, and to associated displacements obeying to a normality rule.

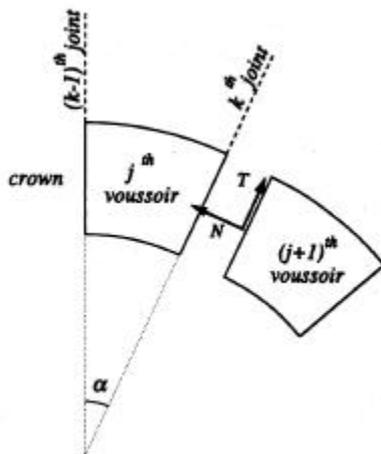


Fig. 1. Joint α and local reference system.

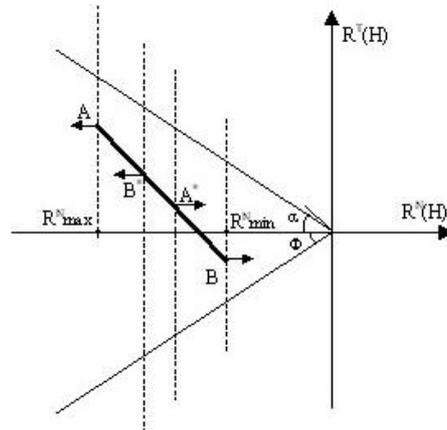


Fig. 2. Drucker's domain, at joint α .

These displacements are relative rotations of the j -th voussoir about either the extrados (point A) or intrados (point B) of the joint \mathbf{a} . However, since the application point of H can vary at the crown from the extrados to the intrados, in dependence of the value of K , points A and B may correspond either to the same or to different application points for H .

For K large enough, H can be applied both at the crown intrados and extrados; thus, inside the segment AB two further points B^* and A^* exist. The range of R^N defined by the segment AA^* corresponds to application point of H at the crown intrados, and the displacements at

points A and A^* are relative rotations of the j -th voussoir about the extrados and the intrados, respectively. On the other hand, the range of R^N defined by the segment B^*B corresponds to application point of H at the crown extrados, and the displacements at points B^* and B are relative rotations of the j -th voussoir about the extrados and the intrados, respectively. Nevertheless, in accordance to the lower bound theorem for standard material, points B^* and A^* guarantees the stability of the joint since they are below the yielding boundary.

By decreasing K sufficiently, both point A and B will correspond to the same application point for H , at the crown extrados; thus, they correspond to relative rotations of the j -th voussoir about the extrados and the intrados of the joint \mathbf{a} , respectively.

An interesting geometric representation of the Drucker's boundary and the associated displacements field at joint \mathbf{a} is that defining two lines of thrust. For K large enough, the first line (point A) touches the arch at the crown intrados and at the joint extrados, while the second line (point B) touches the arch at the crown extrados and at the joint intrados. By decreasing K sufficiently, the first line (point A) will cross the arch at the crown extrados and at the joint extrados, while the second (point B) will cross the arch at the crown extrados and at the joint intrados. Note that due to the symmetry, in any case, both points A and B define a three hinges arch; thus, the normality rule is satisfied since the displacement field is zero and the Drucker's boundary at joint \mathbf{a} is stable.

Drucker's domain for the whole arch can be obtained by taking into account the symmetry and the normality rule: it is the range of R^N common to the Drucker's domains for all the arch joints, that is, for \mathbf{a} ranging from 0° to 90° . The common range for R^N guarantees that the yield surface is reached at the extrados or intrados of the lowest number of joints.

Obviously, if Drucker's domain is empty, the arch is certainly unstable. If Drucker's domain for the arch exists and corresponds to a finite range for H , thus the arch is stable; in that case, a displacement field is associated to each application point of H with hinges - at the most - at three arch joints. By decreasing K , the extension of both local and global Drucker's domain decreases. Collapse condition is reached when the extreme points of the yielding surface of the arch implies the formation of a sufficient number of hinges (five for a symmetric arch subject to its own weight), alternatively located at the intrados and extrados of the arch; therefore, failure corresponds, as well known, to a single value of H applied at the crown extrados and touching the haunches intrados and the springing joints extrados.

The approach proposed above, in the framework of plasticity theory, coincides exactly with that proposed by Coulomb⁴ and integrated by Persy⁶ to investigate the rotational equilibrium of a symmetric masonry arch for friction preventing sliding. It is noteworthy that the Coulomb-Persy approach corresponds to a mixed static-kinematic limit analysis from above, in the sense that the ultimate equilibrium states of the generic joint \mathbf{a} - and then of the whole arch - are always associated to given field displacements obeying a normality rule. This circumstance allowed a modern re-formulation in terms of virtual work by searching from above the limit condition as a function of the parameter K , by identifying simultaneously the collapse mechanism (Sinopoli¹¹).

5 MODIFIED RADENKOVIC'S DOMAIN FOR NON-STANDARD BEHAVIOUR

Assume now a finite value for the friction coefficient $m = \tan f$. Thus, at some joint, the Coulomb yielding boundaries for R^T intersect the segment AB (Fig. 3); Drucker's domain is that for non-standard material. In Fig. 3, segment AC and point B represent the yielding

boundary of the joint; now, the displacements field allows relative tangential displacements. Moreover, a non-associated flow rule characterizes the field displacement; then, bounding theorems for non-standard material need to be applied.

Application of the upper bound theorem gives the Drucker's domain for the same structure made of standard material, that is, without sliding: it is the range of R^N for rotational equilibrium corresponding to the segment AB . Outside this range, the joint made of standard material is unstable and, therefore, also the non-standard material is unstable.

Application of the lower bound theorem should give the Radenkovic's domain; but, in this case, it does not exist, in the sense that it is empty. Nevertheless, a modified Radenkovic's domain can be defined in order to identify the lower bound for stability: it is the range of R^N corresponding to the segment CB (Fig. 4). Note that ultimate states of R^T are defined by the corresponding ultimate states of R^N ; thus, R^T is not an independent function of the thrust H .

Inside the common range of R^N , the joint is made of standard material and, thus, it is stable. The boundary of segment CB is potentially stable; its stability can be verified by using the virtual work theorem and by taking into account the energy dissipated by friction. The same must be made for the states of segment AC to define the joint stable without restrictions.

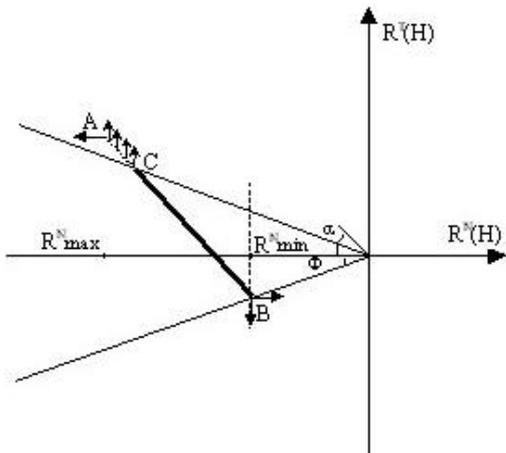


Fig. 3. Drucker's domain for finite friction.

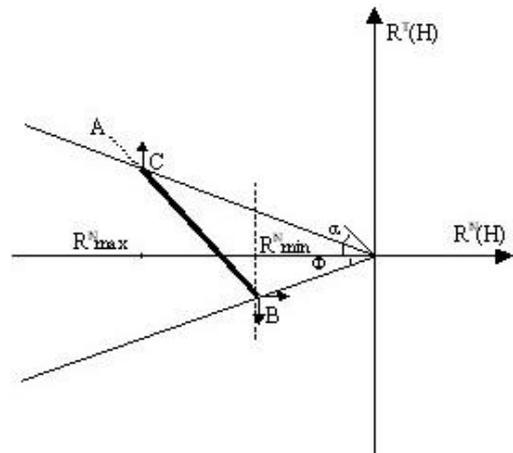


Fig. 4. Modified Radenkovic's domain.

Analyse the displacements field corresponding to the modified Radenkovic's boundary; to this aim, the geometric representation of the yielding boundary can be used. With reference to Fig. 4, boundary points define again two lines of thrust. For H applied at the crown intrados, the first line (point C) crosses at an internal point the joint \mathbf{a} ; while, for H applied at the crown extrados, the second line (point B) touches the intrados of joint \mathbf{a} . These are the first two possible mechanisms. Moreover, note that for K large enough, as it is in the case of Fig. 4 - where $\mathbf{a} > \mathbf{f}$ and the application point of H can vary from the crown intrados to the extrados -, a further line of thrust exists at point C : it touches the joint \mathbf{a} either at the extrados or at an internal point and corresponds to H applied at an internal point of the crown; this is the third possible mechanism.

To have a complete scenario, consider also the case where point B is external to the local

yielding Coulomb domain. Thus, a fourth line of thrust exists: it touches the joint \mathbf{a} either at the intrados or at an internal point, and corresponds to H applied at an internal point of the crown; this is the fourth mechanism, possible only for large values of K . Obviously, all the four mechanisms are impossible at the joint \mathbf{a} ; anyway, when limit analysis of the whole arch is performed, they may correspond to admissible collapse mechanisms.

The modified Radenkovic's domain for the whole arch is obtained as intersection of all the ranges for R^N - locally defined by the modified Radenkovic's domain -, by varying angle \mathbf{a} from 0° to 90° . States internal to the common range for R^N , guarantee the stability of the arch, since they are below Coulomb yielding. Ultimate states lying on the boundary of the global modified Radenkovic's domain are potentially stable; they need to be verified by means of the kinematic theorem.

Note that the global modified Radenkovic's domain is reduced with respect to the global Drucker's domain for standard material. Therefore, global modified Radenkovic' domain defines extreme values of R^N , to which corresponding values of R^T are associated. These extreme values define a reduced range for H , here named H^s ; it is shown in Fig. 5, as a function of the parameter K , for friction coefficient $m=0.364$. In Fig. 5, the range for H of the global Drucker's domain for standard material is also shown; it is the union of the range for H applied at the crown extrados, named H^{e} , and of the range for H applied at the crown intrados, named H^{i} . Inside H^s the arch is stable.

Examine now the stability along the boundary of H^s . For H applied at the crown extrados, the following lines of thrust can be identified:

- For the minimum value of H^s , the line crosses only a sliding rupture joint at some angle \mathbf{a} . This mechanism is impossible.
- For the minimum value of H^s equal to the minimum value of H^{e} , the line crosses at different values of angle \mathbf{a} both a rupture sliding joint and the intrados of haunches. This mechanism also is impossible.
- For the maximum value of H^s , the line crosses only a sliding rupture joint for $\mathbf{a} = 90^\circ$. This mechanism is impossible, unless the maximum value of H^s is equal to the minimum value of H^{e} ; in this case, the line touches also the haunches intrados. This is a collapse mechanism in the range $0.395 \geq m \geq 0.310$, as demonstrated by application of the virtual work theorem. Given m , it can occur for values of K larger than or equal to the critical thickness for standard material; thus, we will name it as *collapse mechanism of the arch of minimum thickness for finite friction and H applied at the crown extrados*.

For H applied at an internal point of the crown joint, the following lines of thrust can be identified:

- For the maximum value of H^s , the line crosses the intrados at some joint and a rupture sliding joint for $\mathbf{a} = 90^\circ$. This mechanism is impossible.
- For the minimum value of H^s , the line crosses the intrados of the sliding rupture joint at some angle \mathbf{a} and the extrados at the springing. This is a possible mechanism and, if admissible, it should correspond to the arch of minimum thickness for finite friction and H applied at an internal point of the crown joint. Note that this is the collapse mechanism first proposed by La Hire³ - revisited then by Mascheroni⁵ - under the assumption of zero

friction at the sliding joint; thus, La Hire³ anticipated, for the sliding joint, the lower Drucker's theorem for non-standard material. However, application of the kinematic theorem demonstrated that this mechanism is not a collapse admissible mechanism since, in correspondence of it, the arch is always stable.

6 BEYOND MODIFIED RADENKOVIC'S BOUNDARY AND CONCLUSIONS

For the stability without restrictions of the arch, it remains to analyze the potentially stable states for H between the boundaries of H^s and the boundaries of $H^{r,e}$ and $H^{r,i}$ (Fig. 5).

By varying the values of friction ($0.395 \geq m \geq 0.310$), the set of the possible mechanisms - even if the sliding rupture joint may correspond to different values of a - contains the possible mechanisms obtained in Sec. 5 for the boundary of H^s , including that of the *arch of minimum thickness for finite friction*; note that this mechanism lies on the left boundary of $H^{r,e}$ (Fig. 5).

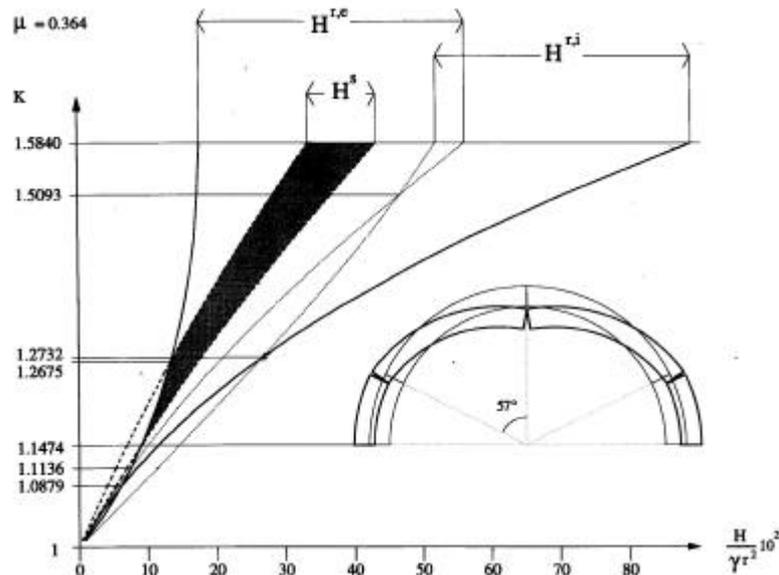


Fig. 5. Standard Drucker and non-standard modified Radenkovic domains for the arch

By excluding other impossible mechanisms since the corresponding displacements field is zero, a new possible mechanism arises for H equal to the maximum value of both $H^{r,i}$ and H^s ; that is, at the boundary of the standard material; we can name it *possible collapse mechanism of minimum thickness for H applied at the extrados*. This situation can occur only for given value of m and for K large enough. However, for an arch subject to its own weight, this is not an admissible collapse mechanism, since, in correspondence of it, the arch is always stable.

In conclusion, we can state that, for an arch subject to its own weight, the possible sliding mechanisms (in the sense that the corresponding displacements field is different from zero) are only: 1) La Hire mechanism; 2) *mechanism of the arch of minimum thickness for H applied at the intrados*; 3) *mechanism of the arch of minimum thickness for H applied at the*

intradados.

Among them, collapse with sliding occurs only for values of m such that $0.395 \geq m \geq 0.310$ and in correspondence to the *mechanism of the arch of minimum thickness for H applied at the intrados*; this collapse requires particular values of both friction coefficient and ratio between the extrados and intrados radius of the arch. Thus, in the presence of usual values of friction, the statement that the collapse of the arch is unlikely assumes a new validity.

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